Received: *December 5, 2000,* Accepted: *January 30, 2001*

Integrability, Seiberg-Witten models and Picard-Fuchs equations

José M. Isidro

Dipartimento di Fisica "G. Galilei" – Istituto Nazionale di Fisica Nucleare
Università degli Studi di Padova
Via Marzolo, 8 — 35131 Padova, Italy, and
Department of Physics, Brandeis University
Waltham, MA 02454-9110, USA
E-mail: `isidro@pd.infn.it`

Abstract: Expanded version of the author's contribution to the Concise Encyclopædia of Supersymmetry, eds. Jonathan Bagger, Steven Duplij and Warren Siegel.

Keywords: Differential and Algebraic Geometry, Integrable Hierarchies.

JHEP01(2001)043

Contents

1. Introduction

Seiberg-Witten (SW) theories [1] have, in their Coulomb branch, a moduli space of physical vacua which coincides with the moduli space of a certain class of Riemann surfaces Σ. In their simplest examples, the latter are hyperelliptic. Such is the situation when one considers a simple, classical gauge group, with or without matter hypermultiplets in the fundamental representation [2]. An essential ingredient is the SW differential λ_{SW} and its period integrals along (a subset of) the homology cycles of Σ. A knowledge of these period integrals amounts to the complete solution of the effective theory, as given by the full quantum prepotential $\mathcal{F}$ [1]. Different approaches have been undertaken in order to compute $\mathcal{F}$ exactly, including instanton corrections [3, 4]. One of them makes use of a set of partial differential equations, with respect to the moduli, satisfied by the periods of λ_{SW} [4]. Such differential equations are second-oder when the matter hypermultiplets are massless, while they become third-order when the masses are non-zero [5]. In either case they can be traced back to a first-order system of differential equations satisfied by the periods of the holomorphic and meromorphic differentials of the second kind, called Picard-Fuchs (PF) equations [6]. It is therefore interesting to study the properties of such a first-order system of differential equations, without regard to their particular physical or mathematical origin [7].

For simplicity we will limit our discussion to the case of hyperelliptic Riemann surfaces. A systematic treatment of instanton corrections for hyperelliptic SW models has been given in [8]; see also [9, 10]. Non-hyperelliptic Riemann surfaces arise naturally in the context of SW models when treated from an M-theoretic point of view [11]. The corresponding instanton corrections have been successfully studied in [12]. Finally, non-hyperelliptic PF equations have also been analysed in [7].

Throughout our study, u_i will denote an arbitrary modulus. Thus, e.g., in the context of SW models, specifying a value for the u_i is equivalent to determining a physical vacuum state of a certain effective $N = 2$ supersymmetric Yang-Mills theory. However, our analysis would hold just as well for the variation of, say, a complex structure on Σ.

2. Hyperelliptic PF equations

Let $p(x)$ be the complex polynomial

$$p(x) = \prod_{l=1}^{2g+1} (x - e_l) = \sum_{j=0}^{2g+1} s_j x^{2g+1-j} , \tag{2.1}$$

where $g \geq 1$. The discriminant Δ of $p(x)$ is defined by

$$\Delta = \prod_{l<n}^{2g+1} (e_l - e_n)^2 . \tag{2.2}$$

Assume that $\Delta \neq 0$. Then the equation

$$y^2 = p(x) \tag{2.3}$$

defines a family of non-singular hyperelliptic Riemann surfaces Σ of genus g. Each one of them is a twofold covering of the Riemann sphere branched over the e_l, plus over the point at infinity. Choices of the e_l such that $\Delta = 0$ will produce singular surfaces. We will assume $\Delta(s_j) \neq 0$ in what follows.

The differential 1-forms on Σ

$$\omega_n = x^n \, \frac{\mathrm{d}x}{y}, \qquad n = 0, 1, 2, \ldots \tag{2.4}$$

are holomorphic for $0 \leq n \leq g - 1$, while they are meromorphic with vanishing residues for all $n \geq g$. Let $\gamma \in H_1(\Sigma)$ be an arbitrary 1-cycle. The period integrals

$$\Omega_n(\gamma) = \int_\gamma \omega_n , \tag{2.5}$$

and the differential equations they satisfy, will be our focus of attention. In order to derive the latter, a generalisation of equations (2.4) and (2.5) is needed. Let $\mathbb{Z}^+$ denote the non-negative integers and $1/2\mathbb{Z}^-$ the negative half-integers. Consider $\mu \in 1/2\mathbb{Z}^-$ and $n \in \mathbb{Z}^+$, and let a given 1-cycle $\gamma \in H_1(\Sigma)$ be fixed. Define the μ-period of x^n along γ, denoted by $\Omega_n^{(\mu)}(\gamma)$, as

$$\Omega_n^{(\mu)}(\gamma) = (-1)^{\mu+1} \Gamma(\mu + 1) \int_\gamma \frac{x^n}{p^{\mu+1}(x)} \, \mathrm{d}x , \tag{2.6}$$

where Γ stands for Euler's gamma function. One can prove that the $\Omega_n^{(\mu)}(\gamma)$ are well defined as a function of the homology class of γ, and that they satisfy the following recursion relations [7]:

$$\Omega_n^{(\mu)}(\gamma) = \frac{1}{n+1-(1+\mu)(2g+1)} \sum_{j=0}^{2g+1} j s_j \Omega_{n+2g+1-j}^{(\mu+1)}(\gamma) \tag{2.7}$$

and

$$\Omega_n^{(\mu+1)}(\gamma) = \frac{1}{n-2g-(1+\mu)(2g+1)} \sum_{j=1}^{2g+1} [(1+\mu)(2g+1-j)-(n-2g)] s_j \Omega_{n-j}^{(\mu+1)}(\gamma). \tag{2.8}$$

We define the *basic range* R to be the set of all $n \in \mathbb{Z}^+$ such that $0 \leq n \leq 2g-1$. Let $Z[s_j]$ be the ring of polynomials in the s_j with complex coefficients. The set of all periods $\Omega_n^{(\mu+1)}(\gamma)$ as n runs over $\mathbb{Z}^+$, when both μ and γ are kept fixed, defines a module over $Z[s_j]$; let it be denoted by $\mathcal{M}^{(\mu+1)}(\gamma)$. Then equations (2.7) and (2.8) prove that $\dim \mathcal{M}^{(\mu+1)}(\gamma) = 2g$. This follows from the observation that equation (2.8) will express any $\Omega_n^{(\mu+1)}(\gamma)$, where $n \geq 2g$, as a certain linear combination of $\Omega_m^{(\mu+1)}(\gamma)$, with $m < n$, and with coefficients that will be certain homogeneous polynomials in the s_j. Repeated application of this recursion will eventually allow to reduce all terms in that linear combination to a sum over the periods of the basic range R. The $\Omega_n^{(\mu+1)}(\gamma)$, where $n \in R$, will be called *basic periods* of $\mathcal{M}^{(\mu+1)}(\gamma)$.

The two modules $\mathcal{M}^{(\mu)}(\gamma)$ and $\mathcal{M}^{(\mu+1)}(\gamma)$ are clearly isomorphic. We can make this isomorphism more explicit as follows. Let $\Omega_n^{(\mu)}(\gamma)$ and $\Omega_m^{(\mu+1)}(\gamma)$, where both n and m run over R, be bases of $\mathcal{M}^{(\mu)}(\gamma)$ and $\mathcal{M}^{(\mu+1)}(\gamma)$, respectively. Arrange them as column vectors $\Omega^{(\mu)} = (\Omega_0^{(\mu)}, \Omega_1^{(\mu)}, \ldots, \Omega_{2g-1}^{(\mu)})^t$ and $\Omega^{(\mu+1)} = (\Omega_0^{(\mu+1)}, \Omega_1^{(\mu+1)}, \ldots, \Omega_{2g-1}^{(\mu+1)})^t$. For notational simplicity we have suppressed the dependence on γ. Now, for every $\mu \in 1/2\mathbb{Z}^-$ there exists a unique matrix $M^{(\mu)}$ such that

$$\Omega^{(\mu)} = M^{(\mu)} \Omega^{(\mu+1)} \,. \tag{2.9}$$

The entries of $M^{(\mu)}$ are certain homogeneous polynomials in the s_j. $M^{(\mu)}$ is non-singular when $\Delta(s_j) \neq 0$. A proof of the statement (2.9) can be found in [7].

As a function of the s_j, we have that $\det M^{(\mu)}$ can only vanish when $\Delta(s_j) = 0$. The entries of $M^{(\mu)}$ are polynomials in the s_j, hence $\det M^{(\mu)}$ will also be a polynomial in the s_j. Decompose $\Delta(s_j)$ into irreducible factors. Up to an overall complex constant, $\det M^{(\mu)}$ must therefore decompose as a product of exactly those same factors present in the decomposition of $\Delta(s_j)$, possibly with different multiplicities (eventually with zero multiplicity, i.e., $\Delta(s_j)$ might have more zeroes than $\det M^{(\mu)}$).

Next we consider derivatives of periods. We have

$$\frac{\partial \Omega_n^{(\mu)}}{\partial s_j} = \Omega_{n+2g+1-j}^{(\mu+1)} \,. \tag{2.10}$$

Now take $n \in R$ in equation (2.10), and use the recursion relation (2.8) as many times as necessary in order to pull the subindex $n+2g+1-j$ back into R. As n runs over R, the right-hand side of equation (2.10) defines the rows of a $(2g \times 2g)$-dimensional matrix, $D_j^{(\mu)}$. In matrix form, equation (2.10) reads

$$\frac{\partial \Omega^{(\mu)}}{\partial s_j} = D_j^{(\mu)}\, \Omega^{(\mu+1)} \,. \tag{2.11}$$

Since equation (2.9) can be inverted when $\Delta(s_j) \neq 0$, we have

$$\frac{\partial \Omega^{(\mu)}}{\partial s_j} = D_j^{(\mu)}\, (M^{(\mu)})^{-1}\, \Omega^{(\mu)} = S_j^{(\mu)}\, \Omega^{(\mu)} \,, \tag{2.12}$$

where we have defined $S_j^{(\mu)} = D_j^{(\mu)}\,(M^{(\mu)})^{-1}$. We finally set $\mu = -1/2$ in order to obtain the PF equations of the hyperelliptic Riemann surface Σ:

$$\frac{\partial}{\partial s_j} \Omega^{(-1/2)} = S_j^{(-1/2)}\, \Omega^{(-1/2)} \,. \tag{2.13}$$

They express the derivatives of the basic periods with respect to the s_j, as certain linear combinations of the same basic periods. The $4g^2$ entries of the matrix $S_j^{(-1/2)}$ are certain rational functions of the s_j, explicitly computable using the recursion relations above. Finally, from a knowledge of the coefficients s_j as functions of the moduli u_i, application of the chain rule and equation (2.13) produces the desired PF equations $\partial\Omega/\partial u_i$.

3. The inverse PF equations and integrability

3.1 The inverse PF equations

Dropping a total derivative under the integral along $\gamma \in H_1(\Sigma)$ we have, from equation (2.3), that

$$n\frac{x^{n-1}}{y} = \frac{x^n}{2y^3}\, p'(x) \,, \qquad n \geq 0 \,. \tag{3.1}$$

Next we define, for any non-negative integer k, the differential λ_k

$$\lambda_k = \frac{x^k}{y}\, p'(x)\, \mathrm{d}x \,, \qquad k \geq 0 \,. \tag{3.2}$$

Modulo total derivatives it holds that

$$\frac{\partial \lambda_k}{\partial s_l} = (-1)^{l+1}\, k\, \frac{x^{k+2g-l}}{y}\, \mathrm{d}x \,. \tag{3.3}$$

It is tempting to call equation (3.3) the *potential property*. In fact it is strongly reminiscent of the property of the SW differential λ_{SW}, that its modular derivatives are holomorphic diferentials on Σ [2]:

$$\frac{\partial \lambda_{SW}}{\partial u_i} = \sum_{j=0}^{g-1} c_i^j \, \omega_j \,, \tag{3.4}$$

where the ω_j are as in equation (2.4), and c_i^j is a constant matrix. We will return to this point in section 3.2. One can now easily prove that the period integrals of λ_k along any $\gamma \in H_1(\Sigma)$ are well-defined functions of the homology class of γ, and that taking modular derivatives does not alter this property. For the second derivatives of λ_k we find

$$\frac{\partial^2 \lambda_k}{\partial s_j \partial s_l} = (-1)^{j+l} \, k \, \frac{x^{k+4g+1-j-l}}{2y^3} \,. \tag{3.5}$$

Then it is easy to prove that the operators

$$\mathcal{L}_l^{(k)} = (2g+1+k-l)\frac{\partial}{\partial s_l} + \sum_{j=0}^{2g+1} (2g+1-j) \, s_j \, \frac{\partial^2}{\partial s_j \partial s_l} \tag{3.6}$$

annihilate the periods of λ_k. The proof follows from equations (3.1), (3.3) and (3.5). With the notation of equation (2.4), the potential property (3.3) reads

$$\frac{\partial \lambda_k}{\partial s_l} = (-1)^{l+1} \, k \, \omega_{2g+k-l} \,, \tag{3.7}$$

and for any fixed value of $k \geq 1$, the basic range R of differentials is spanned by those values of l such that $1 \leq l - k \leq 2g$. If we now denote by Ω_n the period integrals of the above ω_n, then these Ω_n correspond to the $\Omega_n^{(-1/2)}$ of section 2. Finally it is straightforward to establish, using equations (3.6) and (3.7), that the following equations hold:

$$\Omega_{2g+k-l} = \frac{-1}{2g+1+k-l} \sum_{j=0}^{2g+1} (2g+1-j) \, s_j \, \frac{\partial}{\partial s_j} \Omega_{2g+k-l} \,, \tag{3.8}$$

or, equivalently,

$$\Omega_{2g+k-l} = \frac{(-1)^{l+1}}{2g+1+k-l} \sum_{j=0}^{2g+1} (-1)^j \, (2g+1-j) \, s_j \, \frac{\partial}{\partial s_l} \Omega_{2g+k-j} \,. \tag{3.9}$$

3.2 Connection with integrability

Let us summarise our results so far. In section 2 we derived a set of equations for the derivatives of periods as linear combinations (with moduli-dependent coefficients) of periods. Conversely, in section 3.1 we have expressed periods as linear combinations of derivatives of periods.

The purpose of this section, however, goes beyond the triviality of inverting the PF equations (2.13). Rather, we would like to relate the potential property (3.4) of the SW differential with the notion of integrability in the context of SW models [13]–[15]. The integrability of SW models can be traced back to the existence of a prepotential $\mathcal{F}$. In principle, the latter can be obtained exactly by integrating the PF equations. We recall, however, that the PF equations for SW models are at least second-order and that, in their derivation, one makes crucial use of the potential property (3.4) of the SW differential λ_{SW} [5]. In actual SW models, modular derivatives of λ_{SW} need not generate the full g-dimensional space of holomorphic differentials on Σ, the reason being that the Weyl group may select a particular subspace. Once this Weyl symmetry is taken into account, the rank of the matrix c_i^j in equation (3.4) equals the rank of the gauge group.

On the contrary, the PF equations dealt with in sections 2 and 3.1 here are first-order. They apply to *any* hyperelliptic Riemann surface Σ. When written as in section 2, no use is made of the property (3.3). In general, an arbitrary hyperelliptic Riemann surface such as (2.3) does *not* possess a SW differential, in the sense of equation (3.4).

We can however state two necessary and sufficient conditions for the Riemann surface Σ to be integrable, in the sense of SW models. First, an n-dimensional subspace of holomorphic differentials ω_j must be derivable from a *potential* λ. That is, there must exist a (meromorphic) differential λ, and a subset of $m \leq 2g-1$ independent moduli u_l [1], such that the equation

$$\frac{\partial \lambda}{\partial u_l} = \sum_{j=0}^{g-1} d_l^j \omega_j \tag{3.10}$$

holds, for a certain rank-n matrix of constant coefficients d_l^j. Second, when susbtituting equation (3.10) into the first-order PF equations (2.13), the latter must be completely expressible in terms of the u_l. This will lead to a system of PF equations which will be second-order *and* integrable, i.e., it will define a prepotential $\mathcal{F}$.

4. Further applications

The reader may want to consult [17, 18] for nice introductions to SW models. Especially [19] takes the approach based on PF equations. Apart from SW theories, other recent physical contexts in which hyperelliptic period integrals and PF equations play a role include special geometry in $N=2$ supersymmetry [20], mirror symmetry [21, 22], Landau-Ginzburg models, topological field theory and Calabi-Yau manifolds [23], integrability in connection with $N=2$ supersymmetric Yang-Mills theory [24, 25] and Donaldson-Witten theory [26], and the WDVV equation [27, 28]. In the

[1] The moduli space of hyperelliptic Riemann surfaces is $(2g-1)$-dimensional [16].

mathematical literature, PF equations govern the variation of a Hodge structure on a manifold [6]. They also arise in singularity theory [29, 30] and noncommutative geometry [31], in connection with Hilbert's 16th and 21st problems, respectively.

Acknowledgments

It is a great pleasure to thank Diego Bellisai, Gaetano Bertoldi, José D. Edelstein, Kurt Lechner, Pieralberto Marchetti, Marco Matone, João P. Nunes, Paolo Pasti, Howard J. Schnitzer, Dmitri Sorokin, Mario Tonin, Niclas Wyllard and J.-G. Zhou for interesting discussions. The author would like to thank the Department of Physics of Brandeis University, where the final part of this work was completed, for hospitality. This work has been supported by a Fellowship from Istituto Nazionale di Fisica Nucleare (Italy).

References

[1] N. Seiberg and E. Witten, *Electric-magnetic duality, monopole condensation and confinement in $N = 2$ supersymmetric Yang-Mills theory*, *Nucl. Phys.* **B 426** (1994) 19 [hep-th/9407087]; *Monopoles, duality and chiral symmetry breaking in $N = 2$ supersymmetric QCD*, *Nucl. Phys.* **B 431** (1994) 484 [hep-th/9408099].

[2] A. Klemm, *On the geometry behind $N = 2$ supersymmetric effective actions in four dimensions*, [hep-th/9705131].

[3] M. Matone, *Instantons and recursion relations in $N = 2$ Susy gauge theory*, *Phys. Lett.* **B 357** (1995) 342 [hep-th/9506102]; *Koebe 1/4 theorem and inequalities in $N = 2$ super-QCD*, *Phys. Rev.* **D 53** (1996) 7354 [hep-th/9506181]; *Modular invariance and exact wilsonian action of $N = 2$ sym*, *Phys. Rev. Lett.* **78** (1997) 1412 [hep-th/9610204];
G. Bonelli and M. Matone, *Nonperturbative renormalization group equation and beta function in $N = 2$ Susy Yang-Mills*, *Phys. Rev. Lett.* **76** (1996) 4107 [hep-th/9602174]; *Nonperturbative relations in $N = 2$ Susy Yang-Mills and WDVV equation*, *Phys. Rev. Lett.* **77** (1996) 4712 [hep-th/9605090]; *RG flow irreversibility, C-theorem and topological nature of $4-d$ $N = 2$ sym*, *Phys. Rev.* **D 58** (1998) 045006 [hep-th/9712025];
G. Bonelli, M. Matone and M. Tonin, *Solving $N = 2$ sym by reflection symmetry of quantum vacua*, *Phys. Rev.* **D 55** (1997) 6466 [hep-th/9610026];
G. Bertoldi and M. Matone, *Beta function, C-theorem and WDVV equations in 4d $N = 2$ sym*, *Phys. Lett.* **B 425** (1998) 104 [hep-th/9712039]; *$N = 2$ sym RG scale as modulus for WDVV equations*, *Phys. Rev.* **D 57** (1998) 6483 [hep-th/9712109].

[4] A. Klemm, W. Lerche and S. Theisen, *Nonperturbative effective actions of $N = 2$ supersymmetric gauge theories*, *Int. J. Mod. Phys.* **A 11** (1996) 1929 [hep-th/9505150].

[5] J.M. Isidro, A. Mukherjee, J.P. Nunes and H.J. Schnitzer, *A new derivation of the Picard-Fuchs equations for effective $N = 2$ super Yang-Mills theories*, *Nucl. Phys.* **B 492** (1997) 647 [hep-th/9609116]; *On the picard-fuchs equations for massive $N = 2$ Seiberg-Witten theories*, *Nucl. Phys.* **B 502** (1997) 363 [hep-th/9704174]; *A note on the Picard-Fuchs equations for $N = 2$ Seiberg-Witten theories*, *Int. J. Mod. Phys.* **A 13** (1998) 233.

[6] P. Griffiths and J. Harris, *Principles of algebraic geometry*, Wiley Interscience, New York 1978.

[7] J.M. Isidro, *Properties of period integrals of complex algebraic curves*, *J. Geom. Phys.* **29** (1999) 334.

[8] E. D'Hoker, I.M. Krichever and D.H. Phong, *The effective prepotential of $N = 2$ supersymmetric* $\mathrm{SU}(N_c)$ *gauge theories*, *Nucl. Phys.* **B 489** (1997) 179 [hep-th/9609041]; *The effective prepotential of $N = 2$ supersymmetric* $\mathrm{SO}(N_c)$ *and* $\mathrm{Sp}(N_c)$ *gauge theories*, *Nucl. Phys.* **B 489** (1997) 211 [hep-th/9609145];
E. D'Hoker and D.H. Phong, *Strong coupling expansions of* $\mathrm{SU}(N)$ *Seiberg-Witten theory*, *Phys. Lett.* **B 397** (1997) 94 [hep-th/9701055].

[9] D. Bellisai, F. Fucito, M. Matone and G. Travaglini, *Non-holomorphic terms in $N = 2$ Susy wilsonian actions and RG equation*, *Phys. Rev.* **D 56** (1997) 5218 [hep-th/9706099];
D. Bellisai and G. Travaglini, *Instanton calculus and Susy gauge theories on ale manifolds*, *Phys. Rev.* **D 58** (1998) 025008 [hep-th/9712218];
D. Bellisai, F. Fucito, A. Tanzini and G. Travaglini, *Non-perturbative results in global Susy and topological field theories*, hep-th/9812145; *Multi-instantons, supersymmetry and topological field theories*, *Phys. Lett.* **B 480** (2000) 365 [hep-th/0002110]; *Instanton calculus, topological field theories and $N = 2$ super Yang-Mills theories*, *J. High Energy Phys.* **07** (2000) 017 [hep-th/0003272].

[10] J.D. Edelstein, M. Marino and J. Mas, *Whitham hierarchies, instanton corrections and soft supersymmetry breaking in $N = 2$* $\mathrm{SU}(n)$ *super Yang-Mills theory*, *Nucl. Phys.* **B 541** (1999) 671 [hep-th/9805172];
J.D. Edelstein, M. Gomez-Reino and J. Mas, *Instanton corrections in $N = 2$ supersymmetric theories with classical gauge groups and fundamental matter hypermultiplets*, *Nucl. Phys.* **B 561** (1999) 273 [hep-th/9904087];
J.D. Edelstein and J. Mas, *Strong coupling expansion and Seiberg-Witten-Whitham equations*, *Phys. Lett.* **B 452** (1999) 69 [hep-th/9901006].

[11] E. Witten, *Solutions of four-dimensional field theories via M-theory*, *Nucl. Phys.* **B 500** (1997) 3 [hep-th/9703166].

[12] S.G. Naculich, H. Rhedin and H.J. Schnitzer, *One-instanton test of a Seiberg-Witten curve from M-theory: the antisymmetric representation of* $\mathrm{SU}(N)$, *Nucl. Phys.* **B 533** (1998) 275 [hep-th/9804105];

I.P. Ennes, S.G. Naculich, H. Rhedin and H.J. Schnitzer, *One instanton predictions of a Seiberg-Witten curve from M-theory: the symmetric representation of* $\mathrm{SU}(N)$, *Int. J. Mod. Phys.* **A 14** (1999) 301 *One-instanton predictions for non-hyperelliptic curves derived from M-theory*, *Nucl. Phys.* **B 536** (1998) 245 [hep-th/9806144]; *One-instanton predictions of Seiberg-Witten curves for product groups*, *Phys. Lett.* **B 452** (1999) 260 [hep-th/9901124]; *Two antisymmetric hypermultiplets in* $N = 2$ $\mathrm{SU}(N)$ *gauge theory: Seiberg-Witten curve and M-theory interpretation*, *Nucl. Phys.* **B 558** (1999) 41 [hep-th/9904078]; *Tests of M-theory from* $N = 2$ *Seiberg-Witten theory*, hep-th/9911022;

I.P. Ennes, C. Lozano, S.G. Naculich and H.J. Schnitzer, *Elliptic models and M-theory*, *Nucl. Phys.* **B 576** (2000) 313 [hep-th/9912133]; *Seiberg-Witten curves for elliptic models*, hep-th/0007133;

I.P. Ennes, C. Lozano, S.G. Naculich, H. Rhedin and H.J. Schnitzer, *M-theory tested by* $N = 2$ *Seiberg-Witten theory*, [hep-th/0006141].

[13] R. Donagi and E. Witten, *Supersymmetric Yang-Mills theory and integrable systems*, *Nucl. Phys.* **B 460** (1996) 299 [hep-th/9510101].

[14] A. Marshakov, A. Mironov and A. Morozov, *WDVV-like equations in* $N = 2$ *Susy Yang-Mills theory*, *Phys. Lett.* **B 389** (1996) 43 [hep-th/9607109]; *WDVV equations from algebra of forms*, *Mod. Phys. Lett.* **A 12** (1997) 773 [hep-th/9701014]; *More evidence for the WDVV equations in* $N = 2$ *Susy Yang-Mills theories*, *Int. J. Mod. Phys.* **A 15** (2000) 1157 [hep-th/9701123];

A. Morozov, *On integrable structure behind the generalized WDVV equations*, *Phys. Lett.* **B 427** (1998) 93 [hep-th/9711194].

[15] R.W. Carroll, *Prepotentials and Riemann surfaces*, [hep-th/9802130].

[16] M. Schlichenmaier, *An introduction to Riemann surfaces, algebraic curves, and moduli spaces*, Springer-Verlag, Berlin 1989.

[17] L. Alvarez-Gaume and F. Zamora, *Duality in quantum field theory (and string theory)*, [hep-th/9709180].

[18] S.V. Ketov, *Solitons, monopoles and duality: from Sine-Gordon to Seiberg-Witten*, *Fortsch. Phys.* **45** (1997) 237 [hep-th/9611209].

[19] A. Bilal, *Duality in* $N = 2$ *Susy* $\mathrm{SU}(2)$ *Yang-Mills theory: a pedagogical introduction to the work of Seiberg and Witten* [hep-th/9601007].

[20] A. Ceresole, R. D'Auria, S. Ferrara, W. Lerche and J. Louis, *Picard-Fuchs equations and special geometry*, *Int. J. Mod. Phys.* **A 8** (1993) 79 [hep-th/9204035].

[21] S. Hosono, A. Klemm and S. Theisen, *Lectures on mirror symmetry*, [hep-th/9403096].

[22] C. Voisin, *Mirror symmetry* vol. 1 no. 2, SMF/AMS Texts and Monographs 1996.

[23] P. Fré and P. Soriani, *The $N = 2$ wonderland: from Calabi-Yau manifolds to topological field theories*, World Scientific, Singapore 1995.

[24] E. D'Hoker and D.H. Phong, *Seiberg-Witten theory and integrable systems*, [hep-th/9903068]; *Lectures on supersymmetric Yang-Mills theory and integrable systems*, [hep-th/9912271].

[25] A. Lossev, N. Nekrasov and S. Shatashvili, *Testing Seiberg-Witten solution*, [hep-th/9801061].

[26] J.D. Edelstein and J. Mas, *$N = 2$ supersymmetric Yang-Mills theories and Whitham integrable hierarchies*, [hep-th/9902161];
J.D. Edelstein and M. Gomez-Reino, *Integrable hierarchies in Donaldson-Witten and Seiberg-Witten theories*, [hep-th/0010061];
M. Marino, *The uses of Whitham hierarchies*, *Prog. Theor. Phys. Suppl.* **135** (1999) 29;
J.D. Edelstein, M. Gomez-Reino and M. Marino, *Blowup formulae in Donaldson-Witten theory and integrable hierarchies*, [hep-th/0006113].

[27] A. Mironov, *WDVV equations and Seiberg-Witten theory*, [hep-th/9903088].

[28] J.M. Isidro, *On the WDVV equation and M-theory*, *Nucl. Phys.* **B 539** (1999) 379 [hep-th/9805051].

[29] V.I. Arnold, S.M. Gusein-Zade and A.N. Varchenko, *Singularities of differentiable maps*, vol. 2, Birkhäuser, Boston MA 1988.

[30] D. Novikov and S. Yakovenko, *Redundant Picard-Fuchs system for abelian integrals*, [math.DS/0001111].

[31] A. Connes, *Noncommutative geometry*, Academic Press, London 1994; *A short survey of noncommutative geometry*, *J. Math. Phys.* **41** (2000) 3832 [hep-th/0003006].